Assessing the Concordance between Two Georeferenced Variables

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Math-AmSud Project , August 04, 2020

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Outline

- Introduction
- Spatial CCC
- A first Extension
- A Nonparametric Perspective

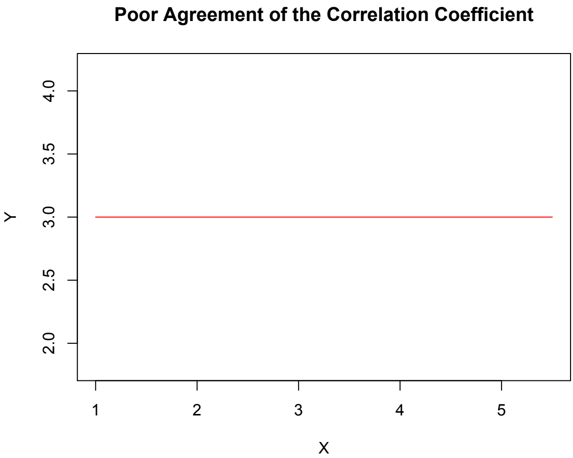
Motivation

- Measurements of agreement are needed to assess the acceptability of new methodology.
- This is important in an assay validation or an instrument validation process.
- Requiring a new measurement to be identical to the truth is often impractical.
- Agreement with continuos measurements (Barnhart: et al., 2007)
  - Descriptive tools
  - Unscaled summary indices based on differences
  - Scaled summary indices attaining values betweem -1 and 1. (A concordance correlation coefficient was studied by Lin (1989)
- How to include spatial information in a concordance coefficient?

Examples

Values for x
$1, 1.5, 2, 2.5, 3,$
$3.5, 4, 4.5, 5, 5.5$

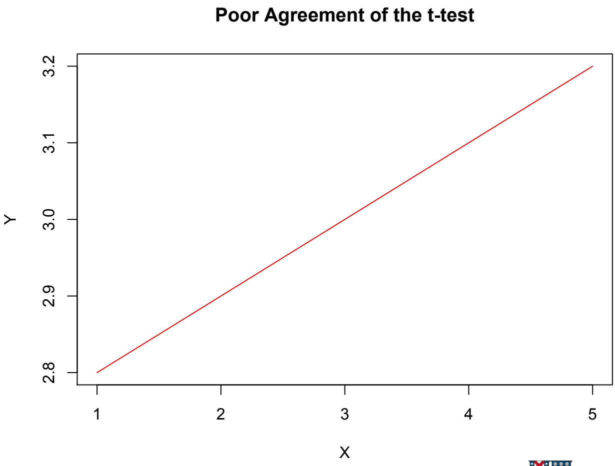
Values for y
$3, 3, 3, 3, 3,$
$3, 3, 3, 3, 3$



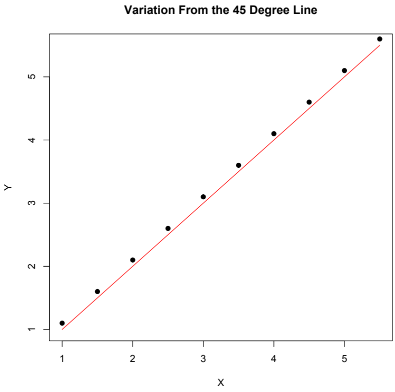
Examples

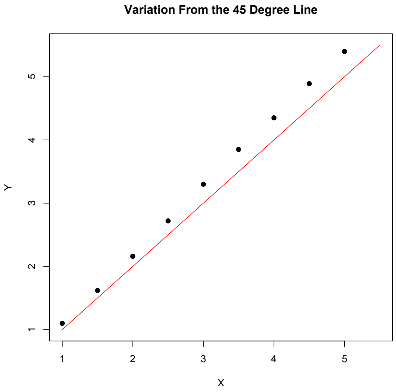
Values for x
$1, 2, 3, 4, 5$

Values for y
$2.8, 2.9, 3, 3.1, 3.2$

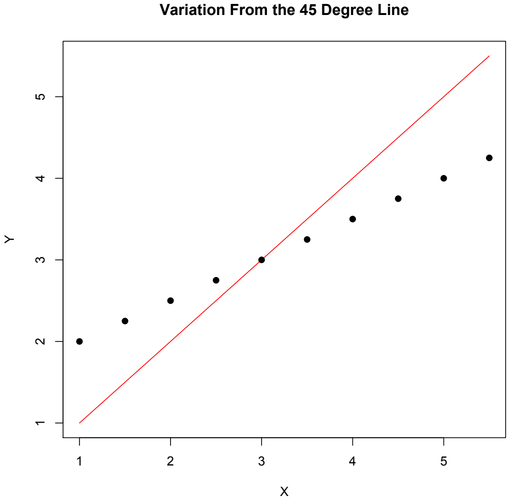


Examples


Variation From the 45 Degree Line
Y
X
1
2
3
4
5


Variation From the 45 Degree Line
Y
X
1
2
3
4
5

Examples



The Concordance Correlation Coefficient

Definition

Assume that the join distribution of X and Y has finite second moment with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and covariance σ_{12}.

- The mean squared deviation of $D = Y - X$ is

$$\begin{aligned}\text{MSD} = \epsilon^2 = \mathbb{E}[D^2] &= \mathbb{E}[(Y-X)^2] \\ &= (\mu_1 - \mu_2)^2 + \sigma_2^2 + \sigma_1^2 - 2\sigma_{21}.\end{aligned}$$

- The Concordance Correlation Coefficient (Lin, 1989) is

$$\rho_c = 1 - \frac{\epsilon^2}{\epsilon^2 | \rho = 0} = \frac{2\sigma_{21}}{\sigma_2^2 + \sigma_1^2 + (\mu_2^2 - \mu_1^2)^2}.$$

The Concordance Correlation Coefficient

Properties

- $\rho_c = \alpha \cdot \rho$, where $\alpha = \dfrac{2}{w + 1/w + v^2}$, $w = \frac{\sigma_2}{\sigma_1}$, $v = \dfrac{\mu_2 - \mu_1}{\sqrt{\sigma_2 \sigma_1}}$, and $\rho = \text{corr}(X, Y)$.
- $|\rho_c| \leq 1$.
- $\rho_c = 0$ if and only if $\rho = 0$.
- $\rho_c = \rho$ if and only if $\sigma_2 = \sigma_1$ and $\mu_2 = \mu_1$.

Sample Concordance

The sample counterpart of ρ_c is given as

$$\widehat{\rho}_c = \frac{2 s_{21}}{s_2^2 + s_1^2 + (\overline{y} - \overline{x})^2}.$$

Our Proposal

- The goal is to construct a concordance coefficient that takes into account the spatial lag $\boldsymbol{h}$, similarly to the variogram and cross-variogram.

- $C_{11}(\boldsymbol{h}) = \mathrm{cov}[X(\boldsymbol{s}), X(\boldsymbol{s}+\boldsymbol{h})]$,
  $C_{22}(\boldsymbol{h}) = \mathrm{cov}[Y(\boldsymbol{s}), Y(\boldsymbol{s}+\boldsymbol{h})]$.
  $C_{12}(\boldsymbol{h}) = \mathrm{cov}[X(\boldsymbol{s}), Y(\boldsymbol{s}+\boldsymbol{h})]$.

- The idea is to define a new coefficient of the form

$$\rho_c = 1 - \frac{\epsilon^2}{\epsilon^2|\rho = 0},$$

but using the above ingredients.

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Our Definition

Definition

Let $(X(\boldsymbol{s}), Y(\boldsymbol{s}))^\top$ be a bivariate second order stationary random field with $\boldsymbol{s} \in \mathbb{R}^2$, mean $(\mu_1, \mu_2)^\top$ and covariance function

$$\boldsymbol{C}(\boldsymbol{h}) = \begin{pmatrix} C_{11}(\boldsymbol{h}) & C_{12}(\boldsymbol{h}) \\ C_{21}(\boldsymbol{h}) & C_{22}(\boldsymbol{h}) \end{pmatrix}.$$

Then the spatial concordance coefficient is defined as

$$\rho_c(\boldsymbol{h}) = \frac{\mathbb{E}[(Y(\boldsymbol{s}+\boldsymbol{h}) - X(\boldsymbol{s}))^2]}{\mathbb{E}[(Y(\boldsymbol{s}+\boldsymbol{h}) - X(\boldsymbol{s}))^2 | C_{12}(\boldsymbol{0}) = 0]} = \frac{2C_{21}(\boldsymbol{h})}{C_{11}(\boldsymbol{0}) + C_{22}(\boldsymbol{0}) + (\mu_1 - \mu_2)^2}.$$

Some Features

- $\rho_c(\boldsymbol{h}) = \eta \cdot \rho_{21}(\boldsymbol{h})$, where $\eta = \dfrac{2\sqrt{C_{11}(\mathbf{0})C_{22}(\mathbf{0})}}{C_{11}(\mathbf{0}) + C_{22}(\mathbf{0}) + (\mu_1 - \mu_2)^2}$.
- $|\rho_c(\boldsymbol{h})| \leq 1$.
- $\rho_c(\boldsymbol{h}) = 0$ iff $\rho_{21}(\boldsymbol{h}) = 0$.
- $\rho_c(\boldsymbol{h}) = \rho_{21}(\boldsymbol{h})$ iff $\mu_1 = \mu_2$ and $C_{11}(\mathbf{0}) = C_{22}(\mathbf{0})$.
- For the nonseparable covariance function

$$C_{ij}(\boldsymbol{h}) = \rho_{ij}\sigma_i\sigma_j R(\boldsymbol{h}, \boldsymbol{\psi}_{ij}),\ \ \rho_{ii} = 1, i, j = 1, 2.$$

where $R(\boldsymbol{h}, \boldsymbol{\psi})$ is a univariate correlation function, we have

$$\rho_c(\boldsymbol{h}) = \eta \cdot \rho_{12},$$

where $\eta = \dfrac{2\sigma_1\sigma_2 R(\boldsymbol{h}, \boldsymbol{\psi}_{12})}{\sigma_1^2 R(\mathbf{0}, \boldsymbol{\psi}_{11}) + \sigma_2^2 R(\mathbf{0}, \boldsymbol{\psi}_{22})}$.

Some Features

- In particular, If $C_{11}(\boldsymbol{h}) = \sigma_1^2 M(\boldsymbol{h}, \nu_1, a_1)$, $C_{22}(\boldsymbol{h}) = \sigma_2^2 M(\boldsymbol{h}, \nu_2, a_2)$, and $C_{21}(\boldsymbol{h}, \nu_{12}, a_{12}) = \rho_{12}\sigma_1\sigma_2 M(\boldsymbol{h}, \nu_{12}, a_{12})$, where $M(\boldsymbol{h}, \nu, a) = \frac{2^{1-\nu}}{\Gamma(\nu)}(a\|h\|)^{\nu} K_{\nu}(a\|h\|)$, and $K_{\nu}(\cdot)$ is a modified Bessel function of the second type and $\rho_{12} = \text{cor}[X(\boldsymbol{s}_i), Y(\boldsymbol{s}_j)]$, Then

$$\rho_c(\boldsymbol{h}) = \frac{2\sigma_1\sigma_2\rho_{12} M(\boldsymbol{h}, \nu_{12}, a_{12})}{\sigma_1^2 + \sigma_2^2} = \eta \cdot \rho_{12},$$

where $\eta = \frac{2\sigma_1\sigma_2 M(\boldsymbol{h}, \nu_{12}, a_{12})}{\sigma_1^2 + \sigma_2^2}$.

- In the previous scheme, if $\nu_{12} = 1/2$, then

$$\rho_c(\boldsymbol{h}) = \eta \cdot \rho_{12},$$

where $\eta = \frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2} e^{-a_{12}||\boldsymbol{h}||}$.

A Local Approach

- Let $(X(\boldsymbol{s}), Y(\boldsymbol{s}))^\top, \boldsymbol{s} \in D \subset \mathbb{Z}^2$ be a bivariate random field.
- Suposse now that D is a finite rectangular grid of $\mathbb{Z}^2$ and that we split D into p subgrids, say $D_i,\ i = 1, \ldots, p$. Then the process $(X_i(\boldsymbol{s}), Y_i(\boldsymbol{s}))^\top, s \in D_i$, represents two subimages defined on D_i.
- Assume that each process $(X_i(\boldsymbol{s}), Y_i(\boldsymbol{s}))^\top$ has a covariance function of the form

$$C^i_{jk}(\boldsymbol{h}) = \left[\rho^i_{jk}\sigma^i_j\sigma^i_k R(\boldsymbol{h}, \boldsymbol{\psi}_i)\right]^2_{j,k=1}, \rho^i_{jk} = 1, i = 1, ...p,\ j, k = 1, 2.$$

- Let $\rho^i_c(\boldsymbol{h})$ be the spatial concordance correlation coefficient of each $(X_i(\boldsymbol{s}), Y_i(\boldsymbol{s}))^\top$. Then

$$\rho^i_c(\boldsymbol{h}) = \frac{2\sigma^i_1\sigma^i_2}{(\sigma^i_1)^2 + (\sigma^i_2)^2}\rho^i_{12} R(\boldsymbol{h}, \boldsymbol{\psi}_i).$$

A Local Approach

- In order to summarize the local concordance coefficients defined for each window, we propose two global concordance coefficients

- $\rho_1(\boldsymbol{h}) = \dfrac{1}{p} \sum_{i=1}^{p} \rho_c^i(\boldsymbol{h})$.
- $\rho_2(\boldsymbol{h}) = \dfrac{2\overline{\sigma}_1\overline{\sigma}_2}{\overline{\sigma}_1^2 + \overline{\sigma}_2^2}\overline{\rho}_{12}R(\boldsymbol{h}, \overline{\boldsymbol{\psi}})$,
  where $\overline{\sigma}_1, \overline{\sigma}_2, \overline{\rho}_{12}, \overline{\boldsymbol{\psi}}$ are the average of the values computed for each sub-image.

- The sample counterparts of $\rho_1(\boldsymbol{h})$ and $\rho_2(\boldsymbol{h})$ are

$$\widehat{\rho}_1(\boldsymbol{h}) = \frac{1}{p}\sum_{i=1}^{p} \widehat{\rho}_c{}^i(\boldsymbol{h}), \quad \widehat{\rho}_2(\boldsymbol{h}) = \frac{2\widehat{\overline{\sigma}}_1\widehat{\overline{\sigma}}_2}{\widehat{\overline{\sigma}}_1^2 + \widehat{\overline{\sigma}}_2^2}\widehat{\overline{\rho}}_{12}R(\boldsymbol{h}, \widehat{\overline{\boldsymbol{\psi}}}).$$

Estimation and Asymptotics

- Let $\left\{Y(\boldsymbol{s}) : \boldsymbol{s} \in D \subset \mathbb{R}^d\right\}$ be a Gaussian random field such that $Y(\cdot)$ is observed on $D_n = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_n\} \subset D$.
- Denote $\boldsymbol{Y} = (Y(\boldsymbol{s}_1), \ldots, Y(\boldsymbol{s}_n))^\top$ and assume that $\mathbb{E}[\boldsymbol{Y}] = \boldsymbol{X}\beta$, cov$(Y(\boldsymbol{t}), Y(\boldsymbol{s})) = \sigma(\boldsymbol{t}, \boldsymbol{s}; \boldsymbol{\theta})$, $\boldsymbol{X}$ is $n \times p$ with rank$(\boldsymbol{X}) = p$, $\boldsymbol{\beta} \in \mathbb{R}^p$ and $\boldsymbol{\theta} \in \mathbb{R}^q$.
- Let $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta})$ be the covariance matrix of $\boldsymbol{Y}$ such that the ij-th element of $\boldsymbol{\Sigma}$ is $\sigma_{ij} = \sigma(\boldsymbol{s}_i, \boldsymbol{s}_j; \boldsymbol{\theta})$.
- The estimation of $\boldsymbol{\theta}$ and β can be made by ML estimation, maximizing

$$L = L(\boldsymbol{\beta}, \boldsymbol{\theta}) = \text{Conts} - \frac{1}{2} \ln \left|\boldsymbol{\Sigma}^{-1}\right| - \frac{1}{2} \left(\boldsymbol{Y} - \boldsymbol{X}\beta\right)^\top \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{Y} - \boldsymbol{X}\beta\right)$$

Estimation and Asymptotics

Theorem

(Mardia and Marshall, 1984) Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\boldsymbol{\Sigma}$, and let those of $\boldsymbol{\Sigma}_i = \dfrac{\partial \boldsymbol{\Sigma}}{\partial \theta_i}$ and $\boldsymbol{\Sigma}_{ij} = \dfrac{\partial^2 \boldsymbol{\Sigma}}{\partial \theta_i \partial \theta_j}$ be λ_k^i and λ_k^{ij}, $k = 1, \dots, n$, such that $|\lambda_1^i| \leq \cdots \leq |\lambda_n^i|$ and $|\lambda_1^{ij}| \leq \cdots \leq |\lambda_n^{ij}|$ for $i, j = 1, \cdots, q$. Suppose that as $n \to \infty$

(i) $\lim \lambda_n = C < \infty$, $\lim |\lambda_n^i| = C_i < \infty$ y $\lim |\lambda_n^{ij}| = C_{ij} < \infty$ for all $i, j = 1, \dots, q$.

(ii) $\|\boldsymbol{\Sigma}_i\|^{-2} = \mathcal{O}(n^{-\frac{1}{2}-\delta})$ for some $\delta > 0$, for $i = 1, \dots, q$.

(iii) For all $i, j = 1, \dots, q$, $a_{ij} = \lim \left[t_{ij}/(t_{ii}t_{jj})^{\frac{1}{2}} \right]$ exists, where $t_{ij} = \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_j\right)$ and $\boldsymbol{A} = (a_{ij})$ is nonsingular.

(iv) $\lim(\boldsymbol{X}^\top \boldsymbol{X})^{-1} = 0$.

Then $(\widehat{\boldsymbol{\beta}}^\top, \widehat{\boldsymbol{\theta}}_n^\top)^\top \xrightarrow{d} \mathcal{N}\left((\boldsymbol{\beta}^\top, \boldsymbol{\theta}^\top)^\top, \boldsymbol{F}_n(\boldsymbol{\theta})^{-1}\right)$, as $n \to \infty$, in an increasing domain sense, where $\boldsymbol{F}_n(\boldsymbol{\theta})$ is the Fisher information matrix of β and $\boldsymbol{\theta}$

Related Work

- Mardia y Marshall (1984) proved that for a Gaussian process with exponential covariance function these conditions are satisfied.
- Acosta and Vallejos (2018) showed that for the covariance (Matérn, $\nu_{12} = n + 1/2$)

$$C_{ij}(\boldsymbol{h}) = \rho_{ij}\sigma_i\sigma_j \exp(-a_{12}\|\boldsymbol{h}\|) \sum_{k=0}^{n} c_k (2a_{12}\|\boldsymbol{h}\|)^{n-k}, \; i = 1, 2,$$

  where $c_k = \frac{(n+k)!}{(2n)!}\binom{n}{k}$, the conditions of the Theorem are satisfied. Here $\boldsymbol{\theta} = (\sigma_1^2, \sigma_2^2, \rho_{12}, a_{12})$.

- Bevilacqua et al. (2015) proved that for a bivariate Gaussian process with the covariance (Matérn, $\nu_{12} = 1/2$)

$$C_{ij}(\boldsymbol{h}) = \rho_{ij}\sigma_i\sigma_j \exp(a_{12}||\boldsymbol{h}||), \; i = 1, 2,$$

  the conditions of the Theorem hold. Here $\boldsymbol{\theta} = (\sigma_1^2, \sigma_2^2, \rho_{12}, a_{12})$.

Asymptotics for the Spatial Concordance

- $\widehat{\rho}^{c}(\boldsymbol{h}) = \widehat{\eta} \cdot \widehat{\rho}_{12}$
- If $\boldsymbol{\theta} = (\sigma_1^2, \sigma_2^2, \boldsymbol{\psi}_{11}^\top, \boldsymbol{\psi}_{22}^\top, \boldsymbol{\psi}_{12}^\top)^\top$. Then $\widehat{\rho}^{c}(\boldsymbol{h}) = g(\widehat{\boldsymbol{\theta}}_n)$.
- The Theorem of Mardia and Marshall (1984) works here for $\widehat{\boldsymbol{\theta}}_n$. The asymptotic normality of $g(\widehat{\boldsymbol{\theta}}_n)$ can be handled via the Delta Method for $g(\cdot)$ differentiable. Indeed,

$$\left(\nabla g(\boldsymbol{\theta})^\top \boldsymbol{F}_n(\boldsymbol{\theta})^{-1} \nabla g(\boldsymbol{\theta})\right)^{-1/2} (g(\widehat{\boldsymbol{\theta}}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} \mathcal{N}(0,1), \ as \ n \to \infty.$$

Computation of the Asymptotic Variance

- For the covariance Matérn, $\nu_{12} = 1/2$,

$$\nabla g(\boldsymbol{\theta})^\top \boldsymbol{F}_n(\boldsymbol{\theta})^{-1} \nabla g(\boldsymbol{\theta}) = \frac{2\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2}\left[\frac{\rho_{12}^2(\sigma_1^2-\sigma_2^2)^2}{(\sigma_1^2+\sigma_2^2)^2 n\boldsymbol{C}} + \frac{2n}{\boldsymbol{C}} + \frac{2\|\boldsymbol{h}\|^2\rho_{12}^2(\rho_{12}^2-1)^2}{n\boldsymbol{C}} + \frac{\rho_{12}^2(\sigma_1^2-\sigma_2^2)^2([\mathrm{tr}(\boldsymbol{B})]^2 - 2\rho_{12}^2\boldsymbol{C})}{(\sigma_1^2+\sigma_2^2)^2}\right] \times \exp\left(-2a_{12}\|\boldsymbol{h}\|\right),$$

- $\boldsymbol{B} = \left(\boldsymbol{R}^{-1}\dfrac{\partial \boldsymbol{R}}{\partial a_{12}}\right),$

- $\boldsymbol{C} = n\mathrm{tr}(\boldsymbol{B}^2) - [\mathrm{tr}(\boldsymbol{B})]^2.$

Hypothesis Testing

- As a consequence of the asymptotic normality, an approximate hypothesis testing problem of the form

$$\mathrm{H}_0 : \rho_c(\boldsymbol{h}) = \rho_0 \text{ versus } \mathrm{H}_1 : \rho_c(\boldsymbol{h}) \neq \rho_0,$$

  can be implemented, for a fixed $\boldsymbol{h}$.

- An approximate confidence interval of the form

$$CI(\rho_c(\boldsymbol{h})) = \widehat{\rho}_c(\boldsymbol{h}) \pm z_{\alpha/2}\sqrt{v},$$

  can be constructed.

- The computation of the variance is a challenging problem for other correlation structures.

- Resampling techniques could be one alternative.

Main Result

Theorem

Let $(X(\boldsymbol{s}), Y(\boldsymbol{s}))^\top$ be a zero mean Gaussian random field with a Wendland-Gneiting bivariate covariance function of the form

$$C_{ij}(\boldsymbol{h}) = \left[\rho_{ij}\sigma_i\sigma_j \left(1 + (\nu+1)\frac{\|\boldsymbol{h}\|}{b_{12}}\right)\left(1 - \frac{\|\boldsymbol{h}\|}{b_{12}}\right)_+^{\nu+1}\right]^2, i,j = 1,2.$$

where $\nu > 0$ is fixed. Then

- $\rho_c(\boldsymbol{h}) = g(\sigma_1^2, \sigma_2^2, \rho_{12}, b_{12}) = \dfrac{2\rho_{12}\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2}\left(1 + (\nu+1)\dfrac{\|\boldsymbol{h}\|}{b_{12}}\right)\left(1 - \dfrac{\|\boldsymbol{h}\|}{b_{12}}\right)_+^{\nu+1}$

- *The ML estimator of $\boldsymbol{\theta}$, $\widehat{\boldsymbol{\theta}}_n$, is asymptotically normal. i.e.,* $\left(\nabla g(\boldsymbol{\theta})^\top \boldsymbol{F}_n(\boldsymbol{\theta})^{-1}\nabla g(\boldsymbol{\theta})\right)^{-1/2}(g(\widehat{\boldsymbol{\theta}}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} \mathcal{N}(0,1),\ as\ n \to \infty.$

An Application

- Two images of size 1600×1200 from Harvard Forest have been considered.

Figure 1: Two images taken from the same site in Harvard Forest, mainly red oak cups. Left: Image taken with an outdoor StarDot NetCam XL 3MP camera. Right: Image taken with an outdoor Axis 223M camera.

- Both images have been transformed to a grey scale and preprocessed.

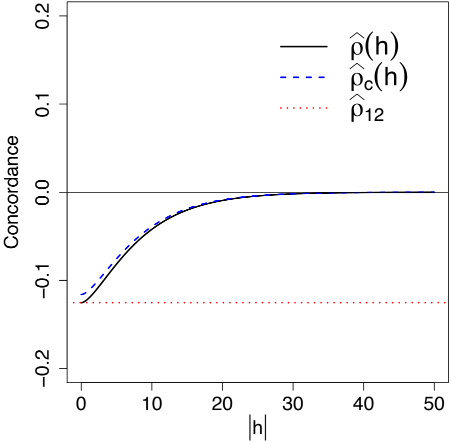
Global Model Fitting

- Lin's concordance: $\widehat{\rho}_c = 0.3177$.
- We consider a Gaussian the random field $(X(\boldsymbol{s}), Y(\boldsymbol{s}))^\top$, $\boldsymbol{s} \in \mathbb{R}^2$.
- We fit a bivariate Matérn covariance model of the form

$$C_{11}(\boldsymbol{h}) = \sigma_1^2 M(\boldsymbol{h}|\nu_1, a_1),$$
$$C_{22}(\boldsymbol{h}) = \sigma_2^2 M(\boldsymbol{h}|\nu_2, a_2),$$
$$C_{12}(\boldsymbol{h}) = C_{21}(\boldsymbol{h}) = \rho_{12}\sigma_1\sigma_2 M(\boldsymbol{h}|\nu_{12}, a_{12}).$$

σ_1^2	σ_2^2	ρ_{12}	ν_1	ν_2	ν_{12}	$1/a_1$	$1/a_2$	$1/a_{12}$
0.04 (0.006)	0.09 (0.037)	-0.12 (0.214)	0.1 (0.015)	0.29 (0.026)	0.99 (2.002)	162.24 (65.973)	162.24 (95.791)	5.64 (2.9589)

Results



$\widehat{\rho}(h)$
$\widehat{\rho}_c(h)$
$\widehat{\rho}_{12}$
Concordance
0.2
0.1
0.0
−0.1
−0.2
0
10
20
30
40
50
|h|

Local Model Fitting

- We splitted the original images into 110 rectangular images of size 19×26.
- We fitted bivariate Matérn covariance functions of the form

$$C_{11}(\boldsymbol{h}) = \sigma_1^2 M(\boldsymbol{h}|\nu_1, a_1),$$
$$C_{22}(\boldsymbol{h}) = \sigma_2^2 M(\boldsymbol{h}|\nu_2, a_2),$$
$$C_{12}(\boldsymbol{h}) = C_{21}(\boldsymbol{h}) = \rho_{12}\sigma_1\sigma_2 M(\boldsymbol{h}|\nu_{12}, a_{12}).$$

for the 110 subimages.

Local Model Fitting

- We compute the global spatial concordance coefficients

$$\widehat{\rho}_1(\boldsymbol{h}) = \frac{1}{p}\sum_{i=1}^{p}\widehat{\rho}_c{}^i(\boldsymbol{h}),$$

and

$$\widehat{\rho}_2(\boldsymbol{h}) = \frac{2\widehat{\overline{\sigma}}_1\widehat{\overline{\sigma}}_2}{\widehat{\overline{\sigma}}_1^2 + \widehat{\overline{\sigma}}_2^2}\widehat{\overline{\rho}}_{12}M(\boldsymbol{h}|\widehat{\overline{\nu}}_{12}, \widehat{\overline{a}}_{12}),$$

where $\overline{\sigma}_1, \widehat{\overline{\sigma}}_2, \widehat{\overline{\rho}}_{12}, \widehat{\overline{\nu}}_{12}$, and $\widehat{\overline{a}}_{12}$ are the average estimations computed using the 110 subimages.

Results

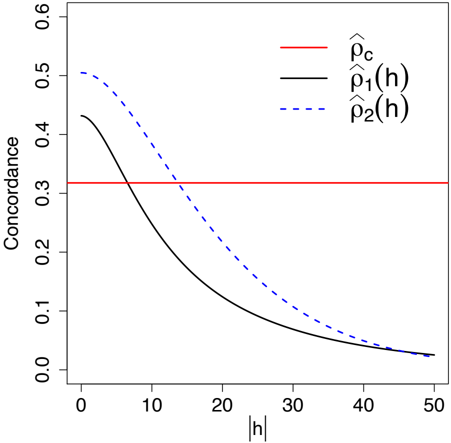


Figure 2: Global concordance coefficients.

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